Gödel's Incompleteness Theorem Part IV – The Diagonal Lemma

Computability and Logic

Part IV - Diagonal Lemma

- The final step in Godel's proof was to prove the Diagonal Lemma:
- For any theory T that contains PA, and for any formula **A(x),** there exists a sentence G such that T $\vDash$ **G** $\leftrightarrow$ **A(g)**, where g is the Gödel number of **G**.
- In other words, for any formula (property) **A(x)**, there is a sentence that says "I have property A(x)"

Proof of Diagonal Lemma I

- The diagonalization of an expression **A(x)** (of L_A) is the expression $\exists$**x (x = a** $\wedge$ **A(x))**, where a is the Gödel number of **A(x)**.
- Consider the function diag(x) = the Gödel number of the expression that is the diagonalization of the expression with Gödel number x.
- It can be easily shown that this is a recursive function.
- Hence, there is a formula $\varphi_{diag}(x,y)$ such that $\varphi_{diag}(m,n)$ is true iff n is the Gödel number of the diagonalization of the expression whose Gödel number is m.
- Moreover, this function is representable in PA, and hence in T, i.e. if diag(m) = n then $T \vDash \forall y\, (\varphi_{diag}(m, y) \leftrightarrow y = n)$

Proof of Diagonal Lemma II

- Let **A(x)** be the formula $\exists$**y (**φ_{diag}**(x, y)** $\wedge$ **B(y))**, with Gödel number a.
- Let G be the diagonalization of **A(x)**, i.e. G is the sentence $\exists$**x (x = a** $\wedge$ $\exists$**y (**φ_{diag}**(x, y)** $\wedge$ **B(y)))**
- So G basically says: “The diagonalization of **A(x)** has property B”.
- But since the diagonalization of **A(x)** is G itself, G ends up saying “I have property B”

Gödel Sentences

- We saw from the Gödel numbering part that for any recursive set of axioms T there is a formula $\varphi_{\mathbf{provable}}(x)$ saying that expression with number x is not provable from axioms T.
- By the Diagonal Lemma, there must therefore be a sentence G such that $T \vDash G \leftrightarrow \neg\varphi_{\mathbf{provable}}(g)$
- This G is called the "Gödel sentence", which basically says "I am not provable from T".

Why any “strong enough” and Sound T is Incomplete

- If G_T is false, then it can be proven from T. But that would mean that T is not sound. Since T is sound, that means that G_T is true. So it is true that G_T is not derivable from T. So, there is a true statement that cannot be derived from T: T is incomplete!